CALCULATION OF DENSITY VARIATION WHICH OCCURS DURING PHASE TRANSFORMATION IN STEFANN'S PROBLEM

(OB UCHETE IZMENENIIA PLOTNOSTI PRI PREVRASHCHENII FAZ V ZADACHE STEFANA)

PMM Vol.25, No.6, 1961, pp. 1143-1144

IU. S. RIAZANTSEV
(Moscow)

(Received July 20, 1961)

Investigations of the distribution of temperature in a medium and of the movement of a phase boundary, when initial and boundary conditions are such that a phase transformation takes place, usually assume that the density of both phases is the same. However, the majority of substances undergo during phase transformation a significant variation in density (several orders greater than the density variation resulting from heat expansion). As examples we offer the densities of the solid ρ_2 and liquid ρ_1 phases of several substances.

	Fe	Cu	Zn	Al	Pb	Ag	H_2O
ρ_2	7.88	8.9	7.19	2.69	11.0	10.6	0.91
ρ_1	6.88	8.21	6.48	2.38	10.6	9.5	1.0

Disregarding this effect can cause inaccuracy in the solution of Stefann's problem. In a number of cases, however, it is not difficult to take into account density variation resulting from phase transformation.

Below we will calculate the density variation during phase transformation for Stefann's problem in its classical presentation (for an example, see [1]).

Assume that the area $x \geqslant 0$ is occupied by a solid phase having a constant temperature $u_2 = c$. Beginning with time $t = 0$, a constant temperature c_1 is maintained on plane $x = 0$. If the melting temperature of the solid phase c_* is such that $c_1 > c_* > c$, then the boundary of the phase transition in the solid phase will begin to be displaced. We will consider that both the liquid and the point of application of the constant temperature can move during the process of melting. Then, if the density of the liquid is less than the density of the solid phase, the area occupied by the liquid will increase with time, not only as a consequence of shifting of the phase boundary due to heat conductivity, but also as a result of expansion.

From the law of conservation of matter, the liquid phase movement velocity caused by the expansion will be equal to $(\rho_2/\rho_1 - 1)\ d\xi/dt$; here, $d\xi/dt$ is the velocity of the phase boundary movement. The task of finding the temperature distribution in the melting and in the solid phase and of determining the phase boundary movement velocity, taking into account the density variation during phase transition, consists in the solution of the equations:

for the liquid phase

$$\frac{\partial u_1}{\partial t} = a_1^2 \frac{\partial^2 u_1}{\partial x^2} + \left(\frac{\rho_2}{\rho_1} - 1\right) \frac{d\xi}{dt} \frac{\partial u_1}{\partial x} \tag{1}$$

for the solid phase

$$\frac{\partial u_2}{\partial t} = a_2^2 \frac{\partial^2 u_2}{\partial x^2} \tag{2}$$

Here, u_1 and u_2 are the temperatures and a_1^2 and a_2^2 the temperature conductivity coefficients of the liquid and solid phases. We must fulfil conditions

$$u_1 = c_1 \quad \text{for } x = -\left(\frac{\rho_2}{\rho_1} - 1\right)\xi, \quad u_2 = c \quad \text{for } t = 0 \tag{3}$$

and melting front conditions

$$u_1 = u_2 = 0, \qquad -k_1 \frac{\partial u_1}{\partial x} + k_2 \frac{\partial u_2}{\partial x} = \lambda \rho_2 \frac{d\xi}{dt} \quad \text{for } x = \xi \tag{4}$$

Here, k_1 and k_2 are the coefficients of the heat conductivity of melting and of the solid phase; λ is the latest heat of melting. It is assumed here that $c_* = 0$; then $c_1 > 0$ and $c < 0$. Obviously, this does not decrease the generality of the analysis.

The system of equations (1) and (2), with conditions (3) and (4), permits a selfsimilar solution for $x/\sqrt{t}$, in which case the law of phase-boundary movement must have the form $\xi = \alpha\sqrt{t}$, where α is a certain constant.

The solution of Equation (1) has the form

$$u_1 = A_1 + B_1 \Phi\left[\frac{x}{2a_1\sqrt{t}} + \left(\frac{\rho_2}{\rho_1} - 1\right)\frac{\alpha}{2a_1}\right]$$

$$\left(\Phi(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-z^2} dz\right)$$

The solution of Equation (2) has the form

$$u_2 = A_2 + B_2\Phi\left[\frac{x}{2a_2\sqrt{t}}\right]$$

Using condition (3), we determine the arbitrary constants

$$A_1 = c_1, \quad B_1 = -\frac{c_1}{\Phi\left[\frac{\alpha\rho_2}{2a_1\rho_1}\right]}, \quad A_2 = -\frac{c_2\Phi\left[\frac{\alpha}{2a_2}\right]}{1-\Phi\left[\frac{\alpha}{2a_2}\right]}, \quad B_2 = \frac{c_2}{1-\Phi\left[\frac{\alpha}{2a_2}\right]}$$

Condition (4) is used to determine the constant a; it assumes the form

$$\frac{k_1c_1}{a_1\Phi\left[\frac{\alpha\rho_2}{2a_1\rho_1}\right]}\exp\left(-\frac{\alpha\rho_2}{2a_1\rho_1}\right)^2 + \frac{k_2c}{a_2\left\{1-\Phi\left[\frac{\alpha}{2a_2}\right]\right\}}\exp\left(-\frac{\alpha}{2a_2}\right)^2 = \frac{\sqrt{\pi}}{2}\lambda\rho_2\alpha \qquad (5)$$

A solution of transcendental equation (5) always exists when $c_1 > 0$ or $c < 0$, since when a changes from 0 to ∞ the left-hand part of the equation changes from ∞ to $-\infty$, the right-hand part from 0 to ∞. Whenever the initial temperature of the solid phase is equal to the melting temperature ($c = 0$), the formulas are simplified. In place of (5) we obtain, for the determination of a, the equation

$$\frac{k_1c_1}{a_1\Phi\left[\frac{\alpha\rho_2}{2a_1\rho_1}\right]}\exp\left(-\frac{\alpha\rho_2}{2a_1\rho_1}\right)^2 = \frac{\sqrt{\pi}}{2}\lambda\rho_2\alpha \qquad (6)$$

The function from a, in the left-hand part of (6), has been tabulated in [1], and the constant a may be determined graphically from (6). Equation (6) differs from the equation for the determination of a in the classical case [1] by the multiple ρ_2/ρ_1. It is not difficult to show that the ratio of the value a, found in (6), to the value found under the assumption $\rho_2 = \rho_1$ is of the same order as the ratio ρ_2/ρ_1.

BIBLIOGRAPHY

1. Tikhonov, A.N. and Samarskii, A.A., *Uravneniia matematicheskoi fiziki (Equations of Mathematical Physics)*. Gostekhteorizdat, 1951.

Translated by C.E.T.